Trigonometry

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1 Angles

1.1 The trigonometric circle

Take an x-axis and an y-axis (orthonormal) and let O be the origin. A circle centered in O and with radius = 1, is called a trigonometric circle or unit circle. Turning counterclockwise is the positive orientation in trigonometry (fig. 1).

1.2 Oriented angles

An angle is the figure formed by two rays that have the same beginning point. That point is called the vertex and the two rays are called the sides of the angle (also legs). If we call [OA the initial side of the angle and [OB the terminal side, then we have an oriented angle. This angle is referred to as ∠AOB and the orientation is indicated by an arrow from the initial side to the terminal side. We can draw the arrow also in the opposite direction, still starting from the initial side of the angle [OA. Both angles represent the same oriented angle. The angle ∠BOA is a different oriented angle which we call the opposite angle of ∠AOB (fig. 2).

Remark: an oriented angle is in fact the set of all angles which can be transformed to each other by a rotation and/or a translation.

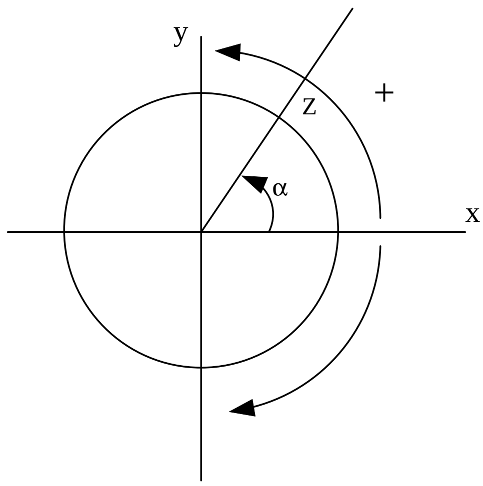


fig. 1 : the trigonometric circle

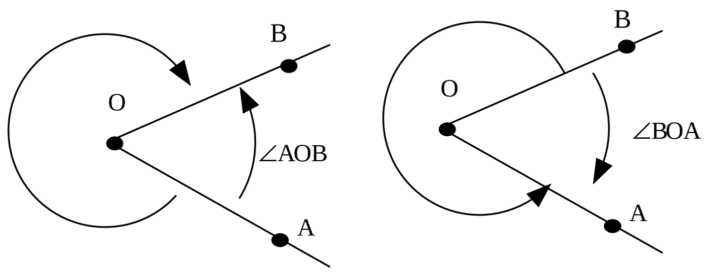


fig. 2 : angles ∠AOB and ∠BOA

The introduction of the trigonometric circle makes it possible to attach a value to each oriented angle ∠AOB, which we will call α from now on. Represent the oriented angle in the trigonometric circle and let the initial side of this angle coincide with the x-axis (see fig. 1). Then the terminal side intersects the trigonometric circle in point Z. Then Z is the representation of the oriented angle α on the trigonometric circle.

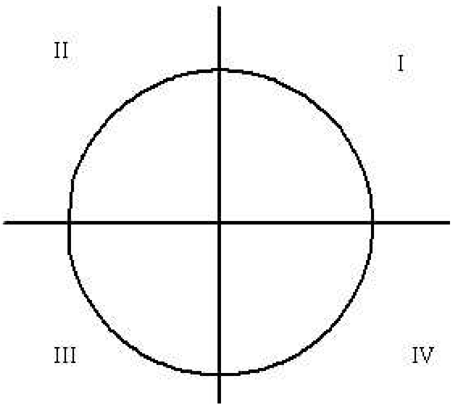


fig. 3 : the four quadrants

If $Z \in$ I: angle α belongs to the first quadrant.

If $Z \in$ II: angle α belongs to the second quadrant.

If $Z \in$ III: angle α belongs to the third quadrant.

If $Z \in$ IV: angle α belongs to the fourth quadrant.

There are two commonly used units of measurement for angles. The more familiar unit of measurement is that of degrees. A circle is divided into 360 equal degrees, so that a right angle is 90°. Each degree is subdivided into 60 minutes and each minute into 60 seconds. The symbols °, ' and " are used for degrees, arcminutes and arcseconds.

In most mathematical work beyond practical geometry, angles are typically measured in radians rather than degrees.

An angle of 1 radian determines on the circle an arc with length the radius of the circle. Because the length of a full circle is $2\pi R$, a circle contains 2π radians. Contrariwise, if one draws in the centre of a circle with radius R an angle of θ radians, then this angle determines an arc on the circle with length $\theta \cdot R$. Subdivisions of radians are written in decimal form.

Next to Z you can put an infinite number of values which differ from each other by an integer multiple of 360° or 2π, because you can make more turns in one or the other direction starting at the initial side of the angle and arriving at the terminal side of the angle (these angles are called coterminal). The set of all these values is called the measure of the oriented angle α. The principal value of α is that value which belongs to $]-180°, 180°]$, resp. $]-\pi,\pi]$.

1.3 Conversion between radians and degrees

Because $2\pi = 360°$,following conversion formulas can be applied:

$$\text{r rad} \rightarrow \left(\frac{360 \bullet \text{r}}{2\pi}\right)^{\circ}$$

$$\text{g}^{\circ} \rightarrow \left(\frac{2\pi \bullet g}{360}\right) \text{rad}$$

Remark: when an angle is represented in radians, one does only mention the value, not the term ‘rad’.

2 The trigonometric numbers

2.1 Definitons

Consider the construction of the oriented angle α as described in the previous paragraph. The terminal side of the angle α intersects the unit circle in the point Z. The $\sin\alpha$ can be defined as the y-coordinate of this point. The $\cos\alpha$ can be defined as the x-coordinate of this point. In this way, we can find the sine or cosine of any real value of α (α ∈ IR). Conversely, the choice of a cosine-value and a sine-value define in the interval [0,2π[only one angle. Overall there are an infinite number of solutions, which one can find by adding on multiples of 2π.

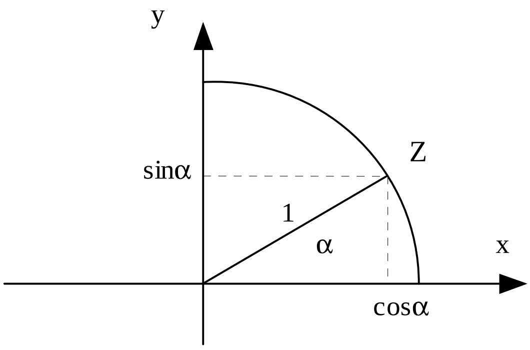


fig. 4 : Sine and cosine in the trigonometric circle

Beside sine and cosine other trigonometric numbers are defined as follows :

tangent : $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

cotangent : $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

secant : $\sec\alpha = \dfrac{1}{\cos\alpha}$

cosecant : $\csc\alpha = \dfrac{1}{\sin\alpha}$

Fig. 5 gives a graphical representation of the above trigonometric numbers in terms of distances associated with the unit circle.

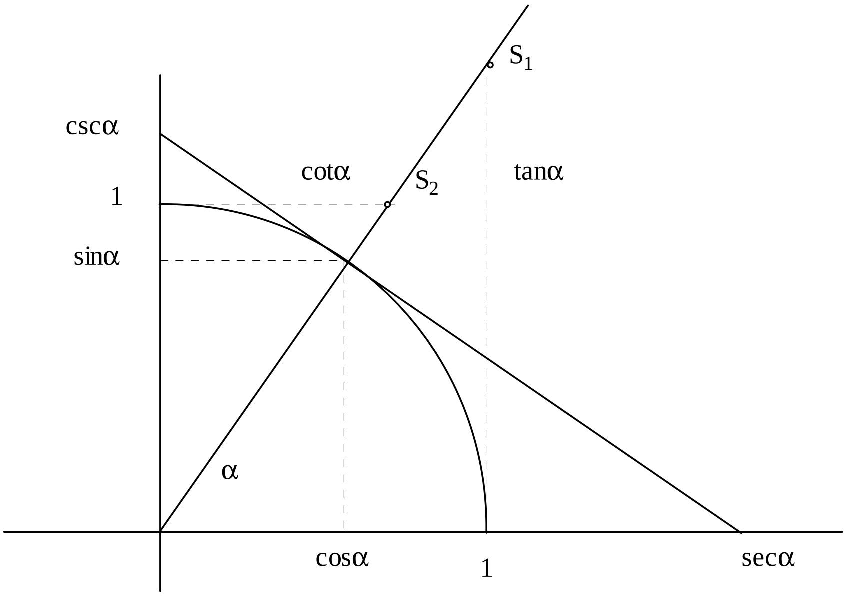


fig. 5 : the graphical representation of the trigonometric numbers in terms of distances associated with the unit circle

Consequently, the trigonometric numbers have values which are in the following areas:

$\sin \alpha \in [-1,1]$ $\cos \alpha \in [-1,1]$

$\tan \alpha \in] -\infty , +\infty [$ $\cot \alpha \in] -\infty , +\infty [$

$\sec \alpha \in] -\infty , -1] \cup [1, +\infty [$ $\csc \alpha \in] -\infty ,-1] \cup [1, +\infty [$

2.2 Some special angles and their trigonometric numbers

α	0	$30° = \pi/6$	$45° = \pi/4$	$60° = \pi/3$	$90° = \pi/2$
$\sin \alpha$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos \alpha$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0
$\tan \alpha$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
$\cot \alpha$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0
$\sec \alpha$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
$\csc \alpha$	∞	2	$\sqrt{2}$	$\sqrt{2}/3$	1

Trigonometric numbers of angles in the other quadrants we shell find through the use of the reference angle (see paragraph 2.6.2.)

2.3 Sign variation for the trigonometric numbers by quadrant

Inside a quadrant the trigonometric numbers keep the same sign (fig. 6).

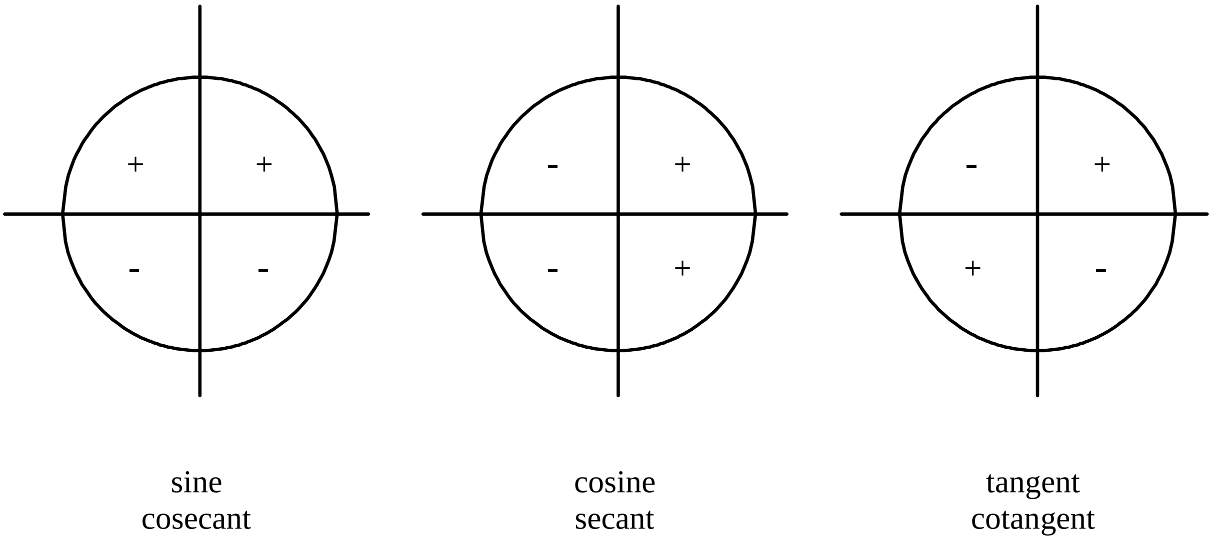


fig.6 : sign variation for the trigonometric numbers by quadrant

2.4 Pythagorean identities

The basic relationship between the sine and the cosine is the Pythagorean or fundamental trigonometric identity: $\cos^2 \alpha + \sin^2 \alpha = 1$

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle (see fig. 7):

$$\|OP\|^2 + \|PZ\|^2 = \|OZ\|^2 \text{ with } \|OP\| = \cos\alpha \quad ; \quad \|PZ\| = \sin\alpha \quad ; \quad \|OZ\| = 1$$

Dividing the Pythagorean identity by either $\cos^2\theta$ or $\sin^2\theta$ yields the following identities:

$1 + \tan^2\alpha = \sec^2\alpha$

$1 + \cot^2\alpha = \csc^2\alpha$

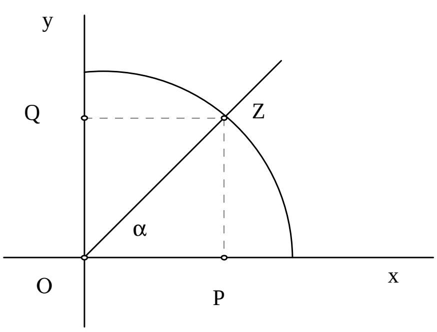


fig. 7 : the triangle OPZ

2.5 Examples

2.5.1 Calculation of the trigonometric numbers

Given: $\sin\alpha = 5/13$

Asked: all other trigonometric numbers

Because the sine of this angle is positive, the angle is situated in the first or second quadrant. We determine the other trigonometric numbers as follows:

- from the Pythagorean trigonometric identity:

  $$\cos^2\alpha = 1 - \sin^2\alpha = 1 - 25/169 = 144/169$$

  we get: $\cos\alpha = \pm\sqrt{144/169} = \pm 12/13$

- $\tan\alpha = \sin\alpha/\cos\alpha = \pm 5/12$
- $\cot\alpha = 1/\tan\alpha = \pm 12/5$
- $\sec\alpha = 1/\cos\alpha = \pm 13/12$
- $\csc\alpha = 1/\sin\alpha = 13/5$

The two possible solutions for some of the trigonometric numbers correspond with the values of these numbers according to the quadrant in which the angle is situated.

Summary :

quadrant	sin	cos	tan	cot	sec	csc
1st	5/13	12/13	5/12	12/5	13/12	13/5
2nd	5/13	-12/13	-5/12	-12/5	-13/12	13/5

2.5.2 Proof the following identity

$$\sec^2 \alpha + \csc^2 \alpha = \sec^2 \alpha \ \ \csc^2 \alpha$$

Proof :

$$\begin{aligned}
\sec^2 \alpha + \csc^2 \alpha &= \frac{1}{\cos^2 \alpha} + \frac{1}{\sin^2 \alpha} \\
&= \frac{\sin^2 \alpha + \cos^2 \alpha}{\cos^2 \alpha \ \sin^2 \alpha} \\
&= \frac{1}{\cos^2 \alpha \ \sin^2 \alpha} \\
&= \sec^2 \alpha \ \csc^2 \alpha
\end{aligned}$$

2.6 Special pairs of angles

The sines, cosines and tangents, cotangents of some angles are equal to the sines, cosines and tangents, cotangents of other angles.

2.6.1 Formulas

a. Supplementary angles (= sum is π)

$\sin(\pi - \alpha) = \sin \alpha$ $\qquad$ $\cos(\pi - \alpha) = - \cos \alpha$

$\tan(\pi - \alpha) = - \tan \alpha$ $\qquad$ $\cot(\pi - \alpha) = - \cot \alpha$

b. Anti-supplementary angles (= difference is π)

$\sin(\pi + \alpha) = - \sin \alpha$ $\qquad$ $\cos(\pi + \alpha) = - \cos \alpha$

$\tan(\pi + \alpha) = \tan \alpha$ $\qquad$ $\cot(\pi + \alpha) = \cot \alpha$

c. Opposite angles (= sum is 2π)

$\sin(2\pi - \alpha) = - \sin \alpha$ $\qquad$ $\cos(2\pi - \alpha) = \cos \alpha$

$\tan(2\pi - \alpha) = - \tan \alpha$ $\qquad$ $\cot(2\pi - \alpha) = - \cot \alpha$

d. Complementary angles (= sum is $\pi / 2$)

$\sin(\pi/2 - \alpha) = \cos \alpha$ $\qquad$ $\cos(\pi/2 - \alpha) = \sin \alpha$

$\tan(\pi/2 - \alpha) = \cot \alpha$ $\qquad$ $\cot(\pi/2 - \alpha) = \tan \alpha$

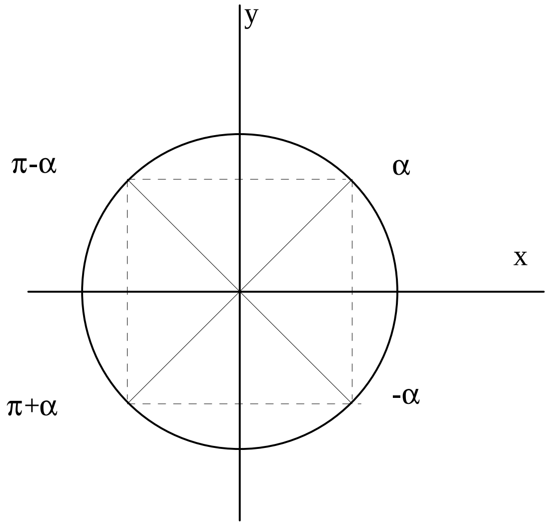


fig. 8 : special pairs of angles

2.6.2 Reference angles

The use of reference angles is a way to simplify the calculation of the trigonometric numbers at various angles.

Associated with every angle drawn in standard position (which means that its vertex is located at the origin and the initial side is on the positive x-axis) (except angles of which the terminal side lies “on” the axes, called quadrantal angles) there is an angle called the reference angle. The reference angle is the acute angle formed by the terminal side of the given angle and the *x*-axis. Angles in quadrant I are their own reference angles. For angles in other quadrants, reference angles are calculated this way:

Quadrant	β (reference angle)
I	$\beta = \alpha$
II	$\beta = \pi - \alpha$
III	$\beta = \alpha - \pi$
IV	$\beta = - \alpha$

The reference angle and the given angle form a pair of angles to which you can apply the properties in the previous paragraph. Due to these properties, the value of a trigonometric number at a given angle is always the same as the value of that angle’s reference angle, except when there is a variation in sign. Because we know the signs of the numbers in different quadrants, we can simplify the calculation of a trigonometric number at any angle to the value of the number at the reference angle for that angle, to be found in the table in paragraph 2.2.

2.6.3 How to find all angles

To find the angle if given a certain trigonometric number, usually there are 2 solutions. Calculators give the most obvious solution, but in practical situations, there can be a second solution, or the second solution can be the only correct solution. In this case the user must adjust the solution given by the calculator.

The following table gives for positive and negative trigonometric numbers the quadrant in which the solution given by the calculator, is situated, and in the last column the quadrant of the second solution:

Input	Calculator	Second solution
Positive sine or cosecant	1	2
Negative sine or cosecant	4	3
Positive cosine or secant	1	4
Negative cosine or secant	2	3
Positive tangent or cotangent	1	3
Negative tangent or cotangent	4	2

2.7 Exercises

2.7.1 Determine for the given trigonometric numbers the other trigonometric numbers; do not determine the angle before.

1. $\sin\alpha = -\sqrt{6}/6$
2. $\csc\alpha = 4/3$
3. $\cot\alpha = -13/6$
4. $\sec\alpha = 25/24$

2.7.2 Proof the following identities

1. $\csc^2\alpha + \cot^2\beta = \csc^2\beta + \cot^2\alpha$

2. $\dfrac{(1-\sin\alpha)(1+\sin\alpha)}{(\sec\alpha+1)(\sec\alpha-1)} = \cos^2\alpha\cot^2\alpha$

3. $\dfrac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha} = (\sec\alpha+\tan\alpha)^2 = \dfrac{1+\sin\alpha}{1-\sin\alpha}$

4. $(1+\cot\alpha)\left(\sec^2\alpha+2\tan\alpha\right) = \dfrac{(1+\tan\alpha)^3}{\tan\alpha}$

2.7.3 Simplify the following expressions by applying the formulas of pairs of angles.

1. $$\frac{\cos\left(\frac{\pi}{2}+x\right)\cos(\pi-x)}{\sin\left(\frac{\pi}{2}-x\right)\sin(x-2\pi)} + \frac{\sin(\pi-x)\cos(\pi+x)}{\sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{3\pi}{2}+x\right)}$$

2. $$\frac{\csc(2\pi+x)\sec(\pi-x)}{\csc\left(\frac{\pi}{2}-x\right)\sec\left(x+\frac{\pi}{2}\right)} - \frac{\sec(2\pi-x)\csc(\pi-x)}{\sec\left(\frac{3\pi}{2}+x\right)\csc\left(\frac{3\pi}{2}-x\right)}$$

2.7.4 Determine the following trigonometric numbers. First find the reference angle, then apply the properties of special pairs of angles.

1. $\sin 120°$
2. $\cos(-135°)$
3. $\tan 225°$
4. $\cot\left(-\frac{3\pi}{4}\right)$
5. $\tan\left(\frac{11\pi}{3}\right)$

2.7.5 Solve in IR. Express the solution(s) in radians.

1. $\cos 5x = -\frac{\sqrt{3}}{2}$
2. $\sin 5x = -\frac{\sqrt{3}}{2}$
3. $\sin 2x = \sqrt{3}\ \sin x$
4. $\sin x = \frac{1}{5}$ and $x \in \left]\frac{\pi}{2}, \pi\right[$;
   asked: $\sin 2x$
5. $2\sin^2 x = 3\cos x$
6. $\tan(3x + 2) = \sqrt{3}$

3 The trigonometric functions

3.1 Periodic functions

<u>Definiton :</u> a function $f : \mathbb{R} \to \mathbb{R}$ is periodic

$$\Leftrightarrow$$

$$\exists\, p \in \mathbb{R}_0 : \forall\, x \in \text{dom } f : x+p \in \text{dom } f \quad \wedge \quad f(x+p) = f(x)$$

If p satisfies this definition, then all positive and negative numbers which are an integer multiple of p also satisfy this definition. Therefore we call the smallest positive number which satisfies this definition the period P of the function. Graphically this periodicity means that the form of the graph of f(x) repeats itself over subsequent intervals with length P.

3.2 Even and odd functions

A function f is called EVEN if:

$$\forall\ x \in \text{dom } f : \quad -x \in \text{dom } f \quad \wedge \quad f(-x) = f(x)$$

Consequently two points with opposite x-values must have the same y-value. So the graph must be symmetric about the y-axis.

A function f is called ODD if:

$$\forall\ x \in \text{dom } f : \quad -x \in \text{dom } f \quad \wedge \quad f(-x) = -f(x)$$

Consequently two points with opposite x-values must have opposite y-values. So the graph is symmetric about the origin.

Remark: we consider the argument of trigonometric functions always in terms of radians.

3.3 Sine function

sin : $\mathbb{R} \to [\,-1,1\,]$: $x \;\to\;$ sin x

The period of this function is 2π. This function is odd, as opposite angles have opposite sines.

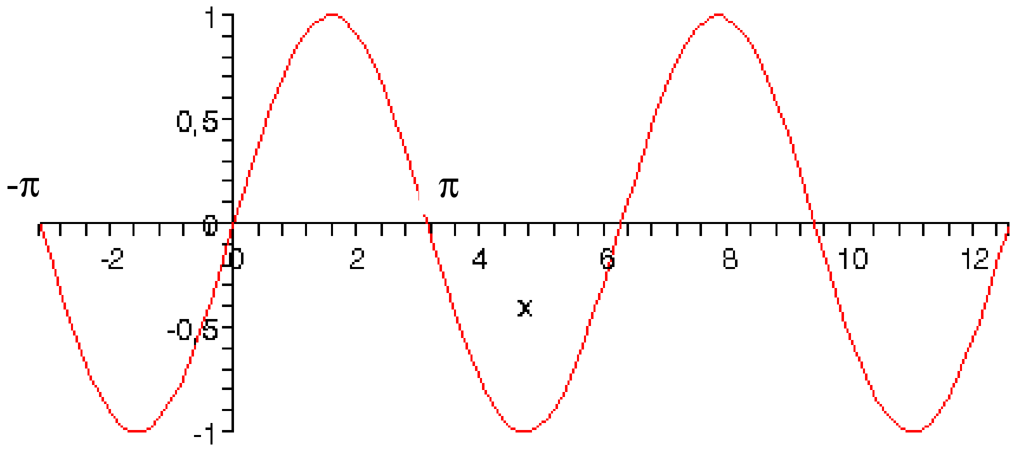


fig. 9 : the sinusoïde

3.4 Cosine function

cos : $\mathbb{R} \to [\,-1,1\,]$: $x \;\to\;$ cos x

The period of this function is 2π. This function is even, as opposite angles have the same cosine.

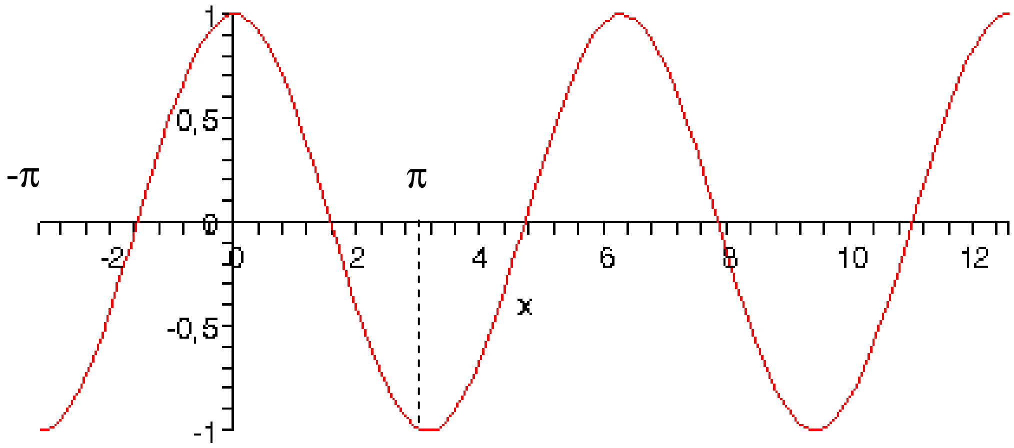


fig. 10 : the cosinusoïde

3.5 Tangent function

$$\tan : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \,,\, k \in \mathbb{Z} \right\} \to \mathrm{IR} \;:\; x \to \tan x$$

The period of this function is π. This function is odd, as opposite angles have opposite tangents.

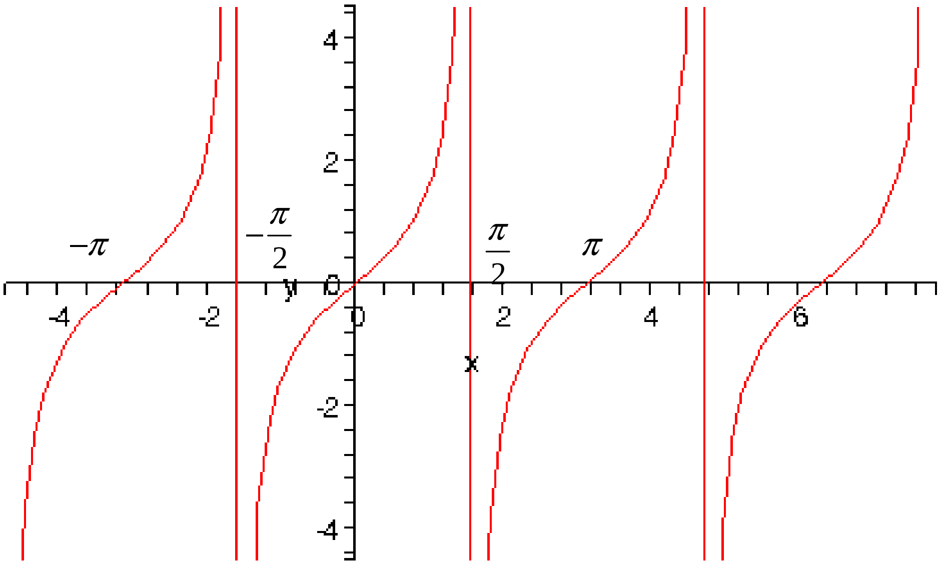


fig. 11 : the tangent function

3.6 Cotangent function

$$\cot : \mathbb{R} \setminus \{ k\pi \,,\, k \in \mathbb{Z} \} \to \mathrm{IR} \;:\; x \to \cot x$$

The period of this function is π. This function is odd, as opposite angles have also opposite cotangents.

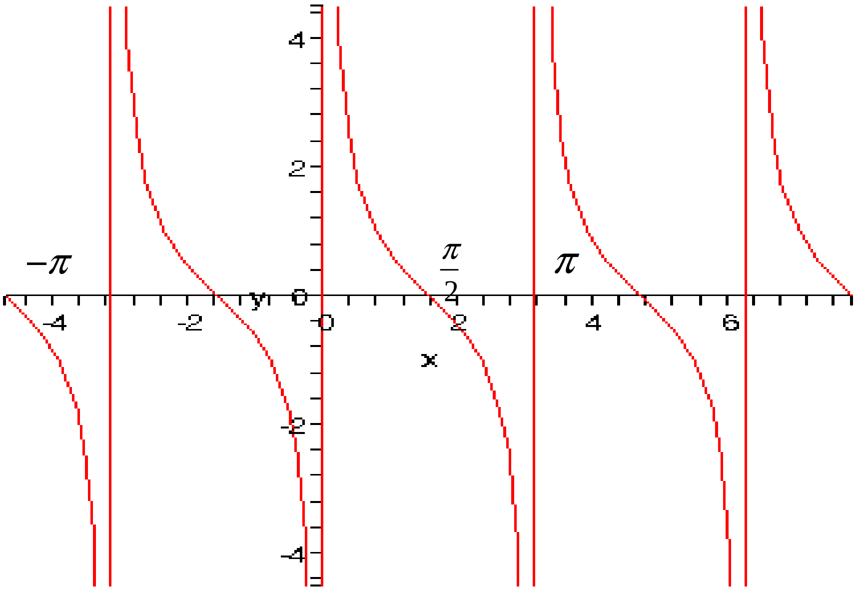


fig. 12 : the cotangent function

3.7 The secant function

$$\sec : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \, , \, k \in \mathbb{Z} \right\} \rightarrow \,]-\infty, -1] \cup [1, +\infty[\; : x \rightarrow \sec x$$

The period of this function is 2π. This function is even, as opposite angles have the same cosines and so the same secants.

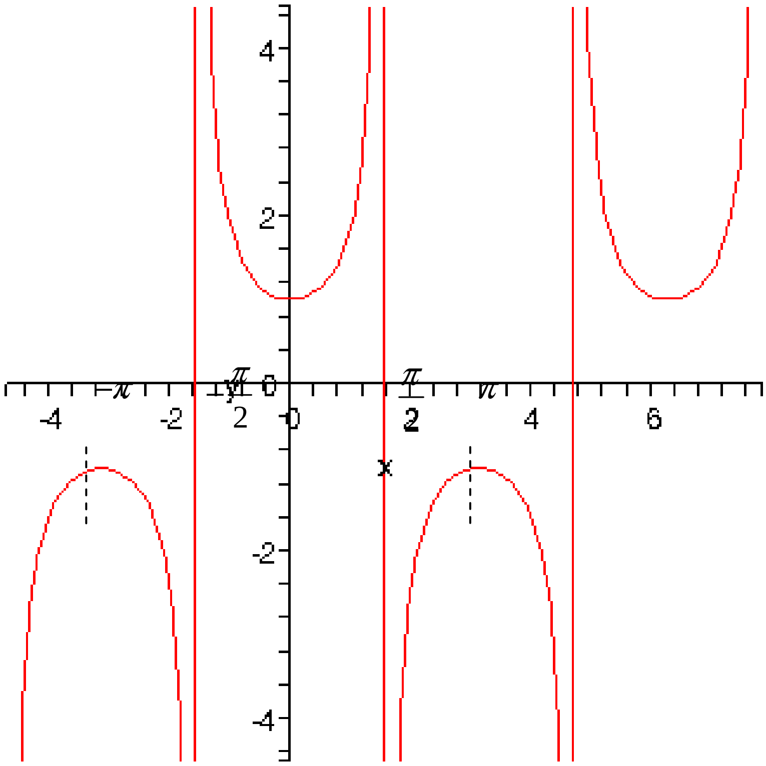


fig. 13 : de secansfunctie

3.8 The cosecant function

$$\csc : \mathbb{R} \setminus \{ k\pi \, , \, k \in \mathbb{Z} \} \rightarrow \,]-\infty, -1] \cup [1, +\infty[\; : x \rightarrow \csc x$$

The period of this function is 2π. This function is odd, as opposite angles have opposite sines and so opposite cosecants.

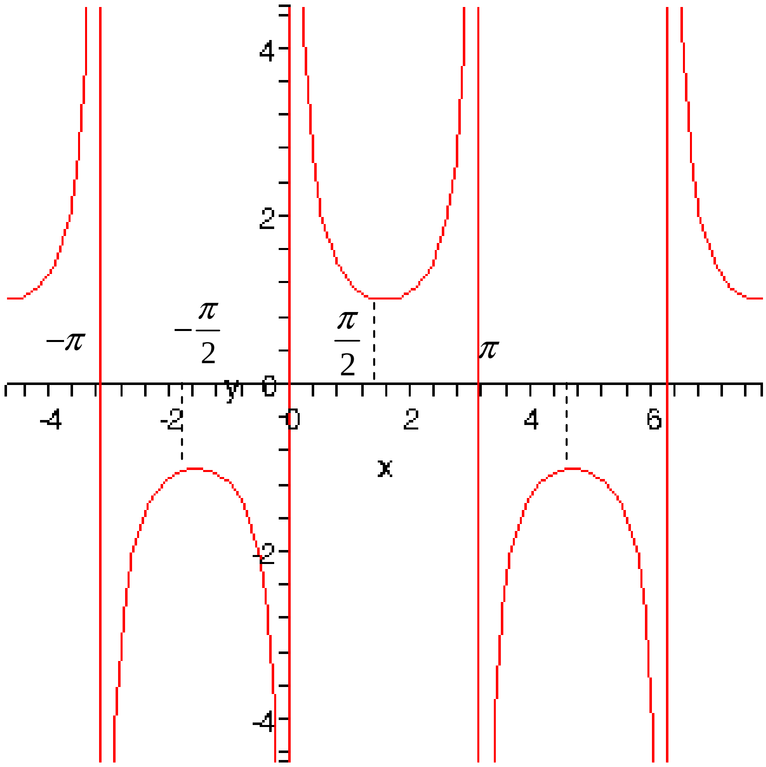


fig. 14 : the cosecant function

3.9 Exercises

3.9.1 Determine the period of the following functions and draw their graph

1. $f(x) = \sin 2x$
2. $f(x) = \cos\left(\frac{x}{3}\right)$
3. $f(x) = \cos\left(\pi + \frac{x}{2}\right)$

4 Right triangles

4.1 Formulas

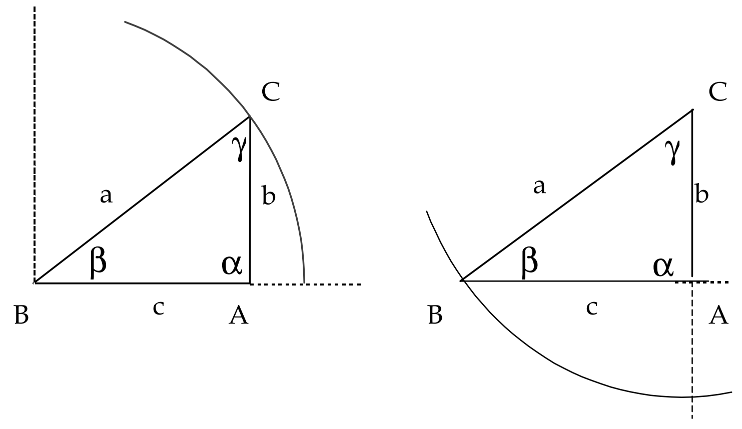


fig. 15 : orthogonal triangles used to set up the formulas in this paragraph

In a right triangle with α as the right angle, the following formulas apply:

$\alpha = \frac{\pi}{2}$ $\qquad \beta + \gamma = \frac{\pi}{2}$ $\qquad a^2 = b^2 + c^2$

If we draw in the triangle above a circle segment with centre in B and radius a (see the first triangle in fig. 15), then we recognize a segment of a circle with radius a. The adjacent side of the right angle c and the opposite side b have resp. the following lengths:

$c = a \cos \beta$ and $b = a \sin \beta$

In a similar way, by considering a circle segment with centre in C and radius a (see second triangle in fig. 15), we find:

$b = a \cos \gamma$ and $c = a \sin \gamma$

In words :

The cosine of an acute angle is the ratio of the length of the adjacent rectangle side and the length of the hypotenuse.

The sine of an acute angle is the ratio of the length of the opposite rectangle side and the length of the hypothenuse.

By division of the first two formulas we get:

$$b = c \tan \beta \qquad \text{or} \qquad c = b \cot \beta$$

If we do the same with the last two formulas, we get:

$$c = b \tan \gamma \qquad \text{or} \qquad b = c \cot \gamma$$

In words :

The tangent of an acute angle is the ratio of the length of the opposite rectangle side and the length of the adjacent rectangle side.

The cotangent of an acute angle is the ratio of the length of the adjacent rectangle side and the length of the opposite rectangle side.

<u>*Example*</u>

Given : $\alpha = 90°$; $\beta = 13°$; $b = 10$

Asked : all missing angles and sides.

Solution:

$\gamma = 90° - \beta = 90° - 13° = 77°$

$$a = \frac{b}{\sin \beta} = \frac{10}{\sin 13°} = 44.5$$

$$c = \sqrt{a^2 - b^2} = 43.5$$

4.2 Exercises

1. Given : ΔABC with a = 45, $\alpha = 90°$, $\beta = 40°10'35''$

   Asked : the remaining sides and angles.

2. An extension ladder stands slantwise to a vertical wall on a horizontal floor. If the ladder is completely extended, it makes an angle of 53°18' with the floor; completely retracted the angle is 29°10', while the top at that moment leans at a heigth of 5 meter against the wall. If we assume that the foot of the ladder does not change, calculate

   - the maximal height one can reach
   - the maximal length of the ladder

3. An incident ray is a ray of light that strikes a surface. The angle between this ray and the perpendicular or normal to the surface is the angle of incidence (α).
   The refracted ray or transmitted ray corresponding to a given incident ray represents the light that is transmitted through the surface. The angle between this ray and the normal is known as the angle of refraction (β).
   The relationship between the angles of incidence and refraction is given by Snell's Law:
   $$\frac{\sin\alpha}{\sin\beta} = n$$

   Example (see figure below):
   A ray of light strikes an air-water interface at an angle of 30 degrees from the normal
   (α = 30°). The relative refractive index for the interface is 4/3. At which distance from P the ray of light hits the bottom if the water is 1 m deep.

   Remark: solve this exercise without calculating the angle β.

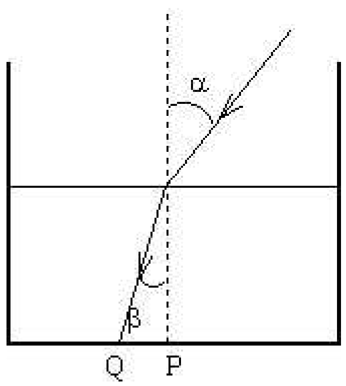


fig. 16 : illustration for exercise 3

In mechanics you will deal with exercises in which forces must be calculated. In the following exercises such situations will be sketched. We will confine to the calculation of angles between bars..

4. Calculate:
   - the angle between FE and the horizontal plane
   - the angle between FC and the vertical plane

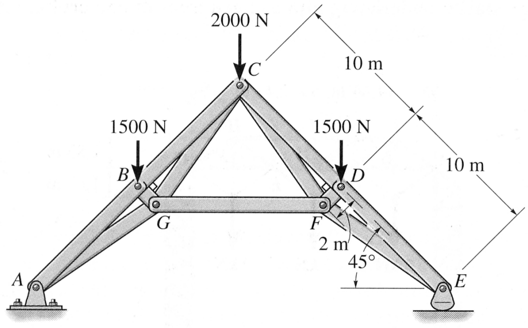


fig. 17 : illustration for exercise 4

5. Calculate the angle between CD and DF

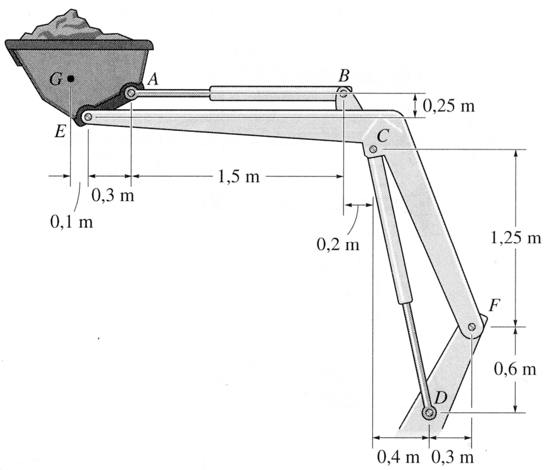


fig. 18 : illustration for exercise 5

6. Calculate the angle between BC and CD

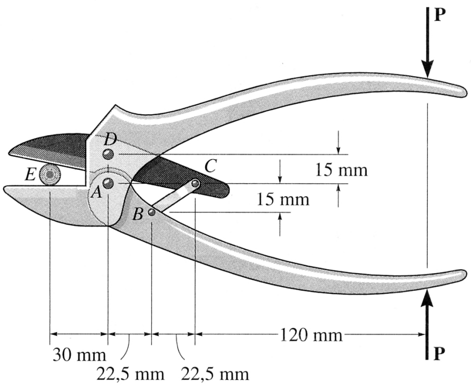


fig. 19 : illustration for exercise 6

5 Oblique triangles

First, remember that also for oblique triangles the sum of angles is 180°.

An oblique triangle is any triangle that is not a right triangle. It could be an acute triangle (all three angles of the triangle are less than right angles) or it could be an obtuse triangle (one of the three angles is greater than a right angle).

Using the formulas for right triangles, we can set up formulas for oblique triangles. Let us consider an oblique triangle ΔABC with sides a, b and c and angles α, β and γ.

5.1 The sine rules

The altitude from A to the opposite side a intersects this side in point S. In this way the triangle is divided in two right angles with one common side AS, with length d. Use now the formulas of a right triangle in triangles ΔABS and ΔACS to calculate d.

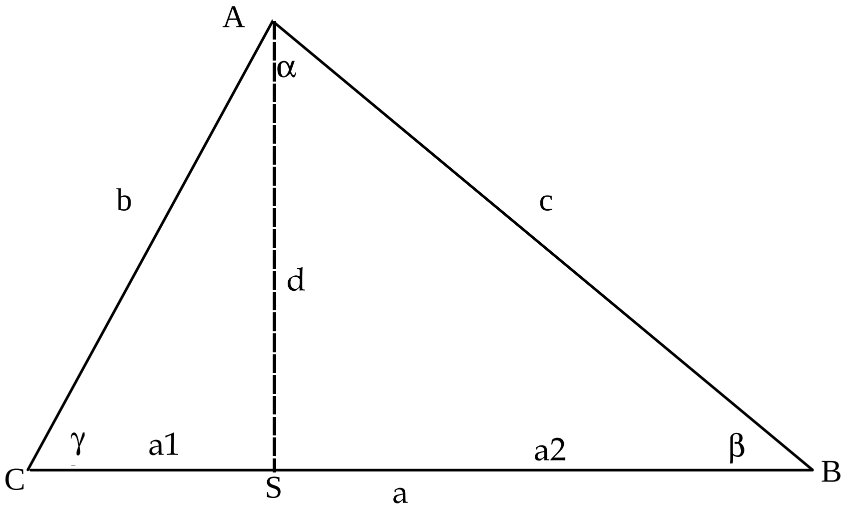


fig. 20 : oblique triangle

$d = c \sin \beta$ and $d = b \sin \gamma$

So we get: $\dfrac{\sin \beta}{b} = \dfrac{\sin \gamma}{c}$

Apply the same reasoning with the altitude from B to the opposite side b to divide the triangle in two right triangles and derive similar formulas in which occur a and the opposite angle α. Then we get:

SINE RULES : $$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

5.2 The cosine rules

This identity can be derived in different ways. In fig. 20 S divides the side a in two parts with length a_1 and a_2. Then a_1 and d can be written respectively as

$a1 = b \cos \gamma$

$d = b \sin \gamma$

In the triangle ΔABS apply Pythagoras theorem:

$$c^2 = d^2 + a_2^2 = d^2 + (a-a_1)^2$$

$$= b^2 \sin^2 \gamma + a^2 + a_1^2 - 2 a a_1$$

$$= b^2 \sin^2 \gamma + a^2 + b^2 \cos^2 \gamma - 2 a b \cos \gamma$$

$$= b^2 + a^2 - 2 a b \cos \gamma$$

The same expression can be derived if S lies outside side a.

Then, similar expressions can be derived for the other angles.

Summarized, in this way we get:

COSINE RULES : $a^2 = b^2 + c^2 - 2 b c \cos \alpha$

$b^2 = a^2 + c^2 - 2 a c \cos \beta$

$c^2 = a^2 + b^2 - 2 a b \cos \gamma$

These statements relate the lengths of the sides of a triangle to the cosine of one of its angles. For example, the first statement states the relationship between the sides of lengths a , b and c, where α denotes the angle contained between sides of lengths *b* and *c* and opposite to the side with length a.

These rules look like the Pythagorean theorem except for the last term, and if you deal with a right triangle, that last term disappears, so these rules are actually a generalization of the Pythagorean theorem.

5.3 Solving oblique triangles

One of the most common applications of the trigonometry is solving triangles – finding missing sides and/or angles, given some information about a triangle. The process of solving triangles can be broken down into a number of cases.

In these situations we will use 3 sorts of formulas, applicable in all triangles:

- the sum of all angles is 180°
- the sine rule : relates two sides to their opposite angles
- the cosine rule : relates the three sides of the triangle to one of the angles.

Naturally, the given information must be such that the given elements allow a triangle:

- the sum of the given angles can not be larger than 180°,
- and the sides must meet the triangle inequality which states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.

a. If you know one angle and the two adjacent sides.

   Then, there is 1 solution:
   you can determine the opposite side by using the cosine rule, another angle by using the sine rule and the remaining angle as 180° minus the two already determined angles.

   Attention: the sine rule gives two solutions for the second angle (supplementary angles). Test the solutions by verifying the properties of a triangle (see exercises).

b. If you know one side and the two adjacent angles.

   Then there is 1 solution:
   the third angle is immediately known as 180° minus the two given angles; the two remaining sides can be determined by using the sine rule.

c. If you know all three sides of a triangle.

   Then there is 1 solution:
   determine one angle by using a cosine rule, the second angle can be determined by using another cosine rule or by using the sine rule. The last angle can be determined by the property of triangles that the sum of all angles must be 180°.

d. If you know sides a and b and β (one of the adjacent angles). In this case, there can be 0, 1 or 2 solutions.

   Determine the angle α by using the sine rule. You will get 0 (if $\sin \alpha > 1$) or 2 solutions (supplementary angles have the same sine). For each solution determine the missing angle γ, and then the length of side c by using the sine rule. Finally you test if each solution which you find is acceptable: you can not have negative angles or sides (see exercises).

5.4 Exercises

1. A tower is seen from the ground under an angle of 21°. If one approaches the tower by 24 meter, then this angle becomes 35°. Determine the height of the tower.

2. Two planes depart from the same point, each in a different direction. The directions form an angle of 32°. The velocity of the first plane is 600 km/hour, the velocity of the second is 900 km/hour. Determine their mutual distance after one hour and a half.

3. The pole of a flag reaches up from a facade with an angle of 45° (see fig. 21). Five meter above the base point of the pole in the wall, there is a cable fixed to the wall with a length of 3,60 meter. At which distance of the base point, measured along the pole, the other end of the cable can be fixed.

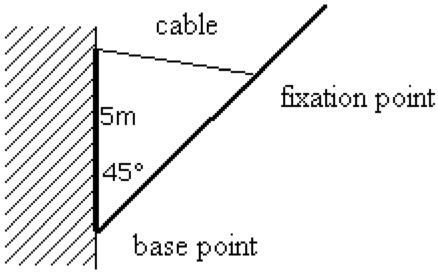


fig. 21 : illustration for exercise 3

4. Solve the previous exercise for a cable with a length of 2 m, respectively with a cable with a length of 8 m.

5. Three observers are at mutual distances of 2, 3 and 4 meters. Determine for each observer the angle under which he sees the other 2 observers.

6. A boat sails north and sees a lighthouse 40° eastwards. After having sailed 20 km, this angle has increased to 80°. Determine at both positions the distance from the boat to the lighthouse.

7. The following figure demonstrates a situation in mechanics. Determine the angle between the ropes AC and AD (d = 1 m).

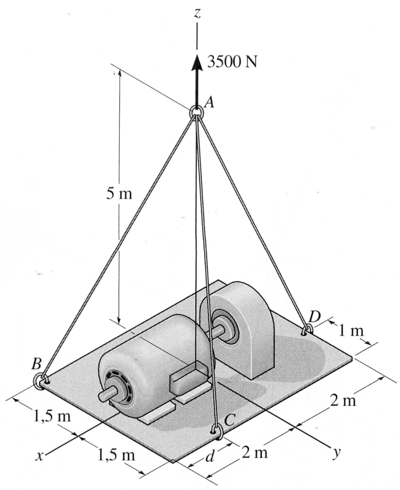


fig. 22 : illustration of exercise 7

6 Extra’s

6.1 Special lines in a triangle

6.1.1 Altitude

An altitude of a triangle is a straight line through a vertex and perpendicular to (i.e. forming a right angle with) the opposite side.

This opposite side is called the base of the altitude, and the point where the altitude intersects the base (or its extension) is called the foot of the altitude. The three altitudes intersect in a single point, called the orthocenter of the triangle. The orthocenter lies inside the triangle if and only if the triangle is acute.

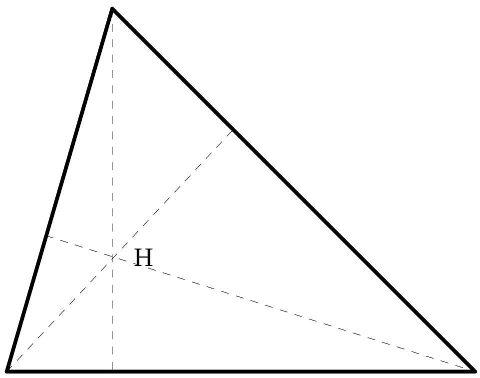


fig. 23 : altitudes

6.1.2 Median

A median of a triangle is a straight line through a vertex and the midpoint of the opposite side, and divides the triangle into two equal areas.

The three medians intersect in a single point, the triangle's centroid. The centroid cuts every median in the ratio 2:1, i.e. the distance between a vertex and the centroid is twice the distance between the centroid and the midpoint of the opposite side.

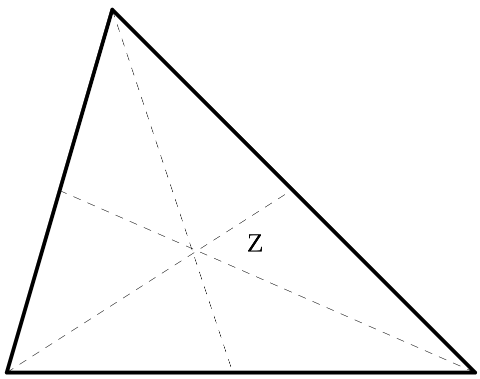


fig. 24 : medians

6.1.3 Other lines

An angle bisector of a triangle is a straight line through a vertex which cuts the corresponding angle in half. The three angle bisectors intersect in a single point, which always lies inside the triangle.

A perpendicular bisector of a side of a triangle is a straight line passing through the midpoint of the side and being perpendicular to it, i.e. forming a right angle with it. The three perpendicular bisectors meet in a single point, the triangle's circumcenter; this point may also lie outside the triangle.

6.2 Isosceles triangles

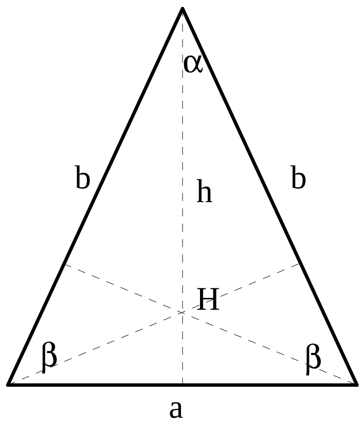


fig. 25 : isosceles triangle

In an isosceles triangle, two sides are equal in length. The unequal side is called its base and the angle opposite the base is called the "vertex angle". The equal sides are called the legs of the triangle. The base angles of an isosceles triangle are always equal.

Property : the altitude and the median from the vertical angle coincide.

Let's call the altitude h, the legs b, the base a, the vertical angle α and the base-angle β:

then : $h = b \sin \beta$ and $\frac{a}{2} = b \cos \beta$

6.2.1 Exercises

1. Set up analog formulas which use the vertex angle.

2. Determine the value of the angles of an isosceles triangle with base 8 and legs 14.

3. Determine the length of the legs of an isosceles triangle with vertex angle 42° and base 12.

4. Determine the length of each side of an isosceles triangle with vertex angle 36° and vertex-altitude 28.

5. In an isosceles triangle with vertex angle 24° the orthocenter lies at a distance of 26 cm to the top. Determine all angles and sides.

6.3 Equilateral triangles

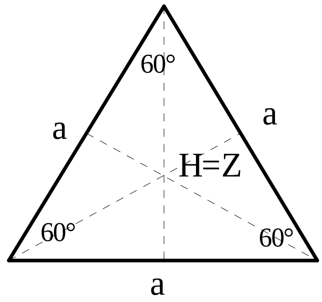


fig. 26 : equilateral triangle

In an equilateral triangle all sides have the same length. Therefore all three angles are equal to each other, and thus 60°.

<u>Property</u> : the altitude from a certain angle coincides with the median from that angle. The orthocenter and the centroid coincide.

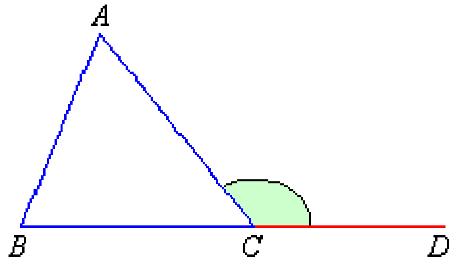
6.3.1 Exercises

1. Determine the distance from the orthocenter/centroid to one of the vertices in terms of the length of the side.
2. Determine the length of an altitude in an equilateral triangle with side 28 cm.
3. The altitude of an equilateral triangle has length 8 cm. Determine the length of the sides.

6.4 Exterior angles

The exterior angle of an angle in a triangle is formed by one side adjacent to that angle and a line extended from the other side adjacent to that angle. Clearly, the exterior angle *ACD* and the adjacent interior angle *ACB* are supplementary. That is:

$$\angle ACD + \angle ACB = 180°$$



The sum of the interior angle and the external angle on the same vertex is 180°. Therefore the sum of all exterior angles is 360° or 2π.

7 Trigonometric formulas

In this paragraph, we discuss formulas involving the trigonometric numbers of a sum or difference of two angles, of a double or half angle, conversions between sums and products of sines and cosines….

As we don't want you to learn these formulas by heart, it is important to understand their mutual connection, the way how one formula can be derived from another formula.

We also want to emphasize that the knowledge of these formulas facilitates solving integrals of trigonometric functions.

7.1 Sum and difference formulas

Let's start with the addition formula for the sine. Then the other formulas can be derived in an easy way.

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ (1)

Replace β by $-\beta$, with $\sin(-\beta) = -\sin \beta$, $\cos(-\beta) = \cos \beta$, then we get:

$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ (2)

For the similar cosine formulas:

$$\begin{aligned} \cos(\alpha + \beta) &= \sin[\frac{\pi}{2} - (\alpha + \beta)] \\ &= \sin[(\frac{\pi}{2} - \alpha) - \beta] \\ &= \sin(\frac{\pi}{2} - \alpha) \cos \beta - \cos(\frac{\pi}{2} - \alpha) \sin \beta \end{aligned}$$

or: $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ (3)

Again, replace β by $-\beta$, then we get:

$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ (4)

You see that the sine formulas keep the plus- or minus sign, but mix the trigonometric functions. The cosine formulas change the sign but hold the trigonometric functions together.

Let's divide (1) side by side by (3), and then divide the nominator and the denominator in the right hand side by (cos α cos β) :

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad (5)$$

And replace β by -β, then we get:

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \qquad (6)$$

7.2 Double-angle formulas

Substitute $\alpha = \beta$ in the previous sum formulas, then we find the double-angle formulas: :

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad (7)$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad (8)$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad (9)$$

Two useful forms of (7) are derived by replacing $\cos^2\alpha$ by $1 - \sin^2\alpha$, resp. $\sin^2\alpha$ by $1 - \cos^2\alpha$:

$$\cos 2\alpha = 1 - 2\sin^2\alpha \qquad (10)$$
$$\cos 2\alpha = 2\cos^2\alpha - 1 \qquad (11)$$

And so:

$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha) \qquad (12)$$

$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha) \qquad (13)$$

7.3 Half-angle formulas

Replace 2α by α in (10) and (11):

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \tag{14}$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \tag{15}$$

7.4 Trigonometric numbers in terms of tan $\alpha/2$

In (8) we divide and multiply the right hand side by $\sec^2\alpha$. By replacing in the denominator $\sec^2\alpha = 1 + \tan^2\alpha$ (see $ 2.4.) and by simplifying the nominator (apply the definition of sec α), we get:

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \tag{16}$$

Replace α by $\frac{\alpha}{2}$:

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{17}$$

By performing the same operation on (7) we find :

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{18}$$

and by replacing α by $\frac{\alpha}{2}$ in (9) , we get:

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \tag{19}$$

7.5 Conversions sum/difference of angles into product of angles and vice versa

In the right hand side of the sum formulas (1) and (2) we notice the same term ($\sin\alpha \cos\beta$). By adding (1) and (2) side by side and bringing factor 2 to the other side, we get :

$$\sin\alpha \ \cos\beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad (20)$$

In a similar way, by subtracting (2) from (1) side by side, we get :

$$\cos\alpha \sin\beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] \qquad (21)$$

By doing the same with formulas (3) and (4) (adding, resp. subtracting side by side), we get:

$$\cos\alpha \cos\beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \qquad (22)$$

$$\sin\alpha \sin\beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad (23)$$

These four formulas convert the product of two cosines and/or sines with a different argument into a sum. The reverse formulas we get by bringing factor ½ to the other side and by substitution:

$$\alpha = \frac{p + q}{2}$$

$$\beta = \frac{p - q}{2}$$

This leads us to the formulas of Simpson:

$$\sin p + \sin q = 2 \sin \frac{p + q}{2} \ \cos \frac{p - q}{2} \qquad (24)$$

$$\sin p - \sin q = 2 \cos \frac{p + q}{2} \ \sin \frac{p - q}{2} \qquad (25)$$

$$\cos p + \cos q = 2 \cos \frac{p + q}{2} \ \cos \frac{p - q}{2} \qquad (26)$$

$$\cos p - \cos q = - 2 \sin \frac{p + q}{2} \ \sin \frac{p - q}{2} \qquad (27)$$

7.6 Exercises

1. Proof the following identity in a triangle:

   $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$

2. Calculate and/or simplify:
   a. $\tan\left(\alpha - \frac{\pi}{4}\right) + \cot\left(\alpha + \frac{\pi}{4}\right)$ $\qquad 0$

   b. $\dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$ $\qquad \tan\left(\alpha - \frac{\pi}{4}\right)$

3. Write in terms of powers of sin α and/or cos α:

   a. $\sin 3\alpha$ $\qquad 3\sin\alpha - 4\sin^3\alpha$

   b. $\cos 4\alpha$ $\qquad 1 - 8\cos^2\alpha + 8\cos^4\alpha$

   c. $\tan\dfrac{\alpha}{2}$ $\qquad \dfrac{\sin\alpha}{\cos\alpha + 1}$

   d. $\dfrac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$ $\qquad \dfrac{1 + \sin\alpha}{\cos\alpha}$

4. Factorize:

   a. $\sin 3\alpha - \sin\alpha$ $\qquad 2\cos 2\alpha\sin\alpha$

   b. $\cos 4\alpha + \cos 5\alpha + \cos 6\alpha$ $\qquad \cos 5\alpha(2\cos\alpha + 1)$

   c. $\tan\alpha - \sin\alpha$ $\qquad 2\tan\alpha\sin^2\dfrac{\alpha}{2}$

   d. $\cos^2\beta - \cos^2\alpha$ $\qquad \sin(\alpha+\beta)\sin(\alpha-\beta)$