Reverse Engineering the Fast Transfer Function

Iain Fox

August 2024

1 Introduction

Let $f : [0,1] \to [0,1]$ be some transfer function. This requires that $f(x)$ be monotone increasing on $[0,1]$ and be differentiable on $(0,1)$.

Let $x = 2^{E-127}\left(1 + 2^{-23}M\right) = 2^{E-127} + 2^{E-150}M$ be a float value such that $114 \le E \le 126$ and $0 \le M \le 2^{23} - 1$ and let $x_u = 2^{23}E + M$ be its 32-bit unsigned integer representation.

2 Finding the Values

Let $\lfloor x \rfloor$ represent the floor of x. The index i will then be equal to

$$\begin{aligned} i &= \left\lfloor 2^{-20}\left(x_u - 114 \cdot 2^{23}\right)\right\rfloor \\ &= \left\lfloor 2^{-20}\left[2^{23}(E - 114) + M\right]\right\rfloor \\ &= \left\lfloor 2^{3}(E - 114) + 2^{-20}M\right\rfloor \\ &= 8(E - 114) + \left\lfloor 2^{-20}M\right\rfloor \end{aligned}$$

This implies that the index is determined by the last 4 bits of the exponent and the first 3 bits of the mantissa. Let B_i and S_i be the bias and scale, respectively, associated with index i.

Let $t = \left\lfloor 2^{8}\left\{2^{-20}M\right\}\right\rfloor$ where $\{x\}$ represents the fractional part of x. This is effectively the 8 bits of the mantissa after the first 3 bits.

$$\begin{aligned} \left\lfloor 255f(x) + \frac{1}{2}\right\rfloor &= \left\lfloor 2^{-7}B_i + 2^{-16}S_i t\right\rfloor \\ &= \left\lfloor 2^{-7}B_i + 2^{-16}S_i\left\lfloor 2^{8}\left\{2^{-20}M\right\}\right\rfloor\right\rfloor \\ 255f(x) + \frac{1}{2} &\approx 2^{-7}B_i + 2^{-16}S_i\left\lfloor 2^{8}\left\{2^{-20}M\right\}\right\rfloor \end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[255f(x)+\frac{1}{2}\right] &\approx \frac{\mathrm{d}}{\mathrm{d}x}\left(2^{-7}B_i+2^{-16}S_i\left\lfloor 2^8\left\{2^{-20}M\right\}\right\rfloor\right)\\
255f'(x) &\approx 2^{-16}S_i\frac{\mathrm{d}}{\mathrm{d}x}\left\lfloor 2^8\left\{2^{-20}M\right\}\right\rfloor\\
&\approx 2^{-8}S_i\frac{\mathrm{d}}{\mathrm{d}x}\left\{2^{-20}M\right\}\\
&\approx 2^{-28}S_i\frac{\mathrm{d}M}{\mathrm{d}x}
\end{aligned}$$

Now, setting $M = 127\cdot 2^{12}$ makes t lie in the middle of its possible values to minimize error. It is also, then, the case that E is constant for t near this value of M.

$$\begin{aligned}
x &= 2^{E-127}\left(1+2^{-23}M\right)\\
\frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[2^{E-127}\left(1+2^{-23}M\right)\right]\\
1 &= 2^{E-127}\frac{\mathrm{d}}{\mathrm{d}x}\left(1+2^{-23}M\right) && \textbf{E is constant.}\\
&= 2^{E-150}\frac{\mathrm{d}M}{\mathrm{d}x}\\
\frac{\mathrm{d}M}{\mathrm{d}x} &= 2^{150-E}
\end{aligned}$$

Plugging this into our derivative approximation gives that

$$\begin{aligned}
255f'(x) &\approx 2^{-28}S_i 2^{150-E}\\
&\approx 2^{122-E}S_i\\
S_i &\approx 255\cdot 2^{E-122}f'(x)
\end{aligned}$$

Returning to the original approximation now gives us that

$$\begin{aligned}
255f(x)+\frac{1}{2} &\approx 2^{-7}B_i+2^{-16}S_i\left\lfloor 2^8\left\{2^{-20}M\right\}\right\rfloor\\
&\approx 2^{-7}B_i+2^{-16}S_i\left\lfloor 2^8\left\{127\cdot 2^{-8}\right\}\right\rfloor\\
&\approx 2^{-7}B_i+127\cdot 2^{-16}S_i\\
2^{-7}B_i &\approx 255f(x)+\frac{1}{2}-127\cdot 2^{-16}S_i\\
B_i &\approx 255\cdot 2^7f(x)+2^6-127\cdot 2^{-9}S_i
\end{aligned}$$