Matrix Minors

We can derive two distinct $O(n^3)$ algorithms to solve this problem.

Fundamentally, both of them start from the fact that, as follows from Cramer's rule, we can construct the adjugate matrix $(\operatorname{adj} A)_{ij} = (-1)^{i+j} M_{ji}$, for which it will hold that $A \operatorname{adj} A = I_n \det A$.

When A is invertible, it means that we can find $\operatorname{adj} A$ as $A^{-1} \det A$. But what if the matrix is not invertible?

Cayley-Hamilton theorem. One way to approach this is via the Cayley-Hamilton theorem, which says that the matrix is a root of its own characteristic polynomial $p_A(x) = \det(A - xI)$.

If we consider its coefficients $p_A(x) = c_0 + c_1 x + \dots + c_n x^n$, we should notice that $c_0 = \det A$. Thus, we can rewrite it in the form

$$P_A(A) = \det A + A(c_1 + c_2 A + \dots + c_n A^{n-1}) = 0.$$

From this, when A is invertible, we can express $\operatorname{adj} A$ as

$$\operatorname{adj} A = -(c_1 + c_2 A + \dots + c_n A^{n-1}),$$

then it will follow from the continuity argument that the same is actually true for singular matrices. This already provides us with one solution, as we can both find and apply the characteristic polynomial of the matrix in $O(n^3)$ using the Frobenius normal form.

Okay then, but what if we can't realistically do this without any references within the span of the contest? Well, there is also an approach with a much simpler implementation!

Schur complement. Let k be the rank of the matrix. If $k = n$, we can simply find the inverse matrix and multiply it with the determinant. On the other hand, if $k < n - 1$, all matrix minors are simply zero, so the only non-trivial case in this problem is when $k = n - 1$.

Let's embed our matrix A into an $(n+1) \times (n+1)$ matrix that *is* invertible:

$$A' = \begin{bmatrix} A & u \\ v^\top & x \end{bmatrix},$$

where u is a column, $v^\top$ is a row, and x is a scalar, selected in such a way that $\det A' \neq 0$. Let's say that

$$(A')^{-1} = \begin{bmatrix} \hat{A} & \hat{u} \\ \hat{v}^\top & \hat{x} \end{bmatrix}$$

Then, we can rewrite

$$A'(A')^{-1} = \begin{bmatrix} A\hat{A} + u\hat{v}^\top & A\hat{u} + u\hat{x} \\ v^\top \hat{A} + x\hat{v}^\top & v^\top \hat{u} + x\hat{x} \end{bmatrix} = I_{n+1}$$

Using the first row, we arrive at

$$\begin{cases} A\hat{A} + u\hat{v}^\top &= I_n, \\ A\hat{u} + u\hat{x} &= 0. \end{cases}$$

From the second equation, we express $u\hat{v}^\top = -A\frac{\hat{u}\hat{v}^\top}{\hat{x}}$, and placing it in the first we arrive at

$$A\left(\hat{A} - \frac{\hat{u}\hat{v}^\top}{\hat{x}}\right) = I_n$$

Due to the connection between the inverse matrix and minors, we get $\hat{x} = \frac{\det A}{\det A'}$, from which we get

$$A(\hat{x}\hat{A} - \hat{u}\hat{v})\det A' = I_n \det A,$$

therefore we get $\operatorname{adj} A = (\hat{x}\hat{A} - \hat{u}\hat{v}^\top)\det A'$, which is true for all matrices due to continuity. Thus, we can find $\operatorname{adj} A$ by finding appropriate A' (e.g. by using random u and $v^\top$), and then getting its inverse in $O(n^3)$ with Gaussian elimination.

In all this, the expression $\hat{A} - \frac{\hat{u}\hat{v}^\top}{\hat{x}}$ is known as the Schur complement of $\hat{x}$ in A'.